1. (1 Punkt)

weil physikalische Größen nur reelle Werte annehmen können (1)

2. (1 Punkt)

$H\Psi = E\Psi$ (1)

3. (1 Punkt)

$\Psi(x,t) = \Psi(x)\,\mathrm{e}^{-\mathrm{i}Et/\hbar}$ (1)

4. (2 Punkte)

(a) nichts, weil $\Delta x = \infty$ (1)

(b) $E = p_x^2/2m$ (1)

5. (6 Punkte)

(a)

$$\begin{aligned} H\Psi_{l,n}(x,y) &= (H_x + H_y)\,\phi_l(x)\,\phi_n(y) \\ &= \phi_n(y)\,H_x\phi_l(x) + \phi_l(x)\,H_y\phi_n(y) \\ &= \phi_n(y)\,E_l\phi_l(x) + \phi_l(x)\,E_n\phi_n(y) \\ &= (E_l + E_n)\,\phi_l(x)\,\phi_n(y) \\ &= (E_l + E_n)\,\Psi_{l,n}(x,y) \end{aligned}$$ (1)

Eigenwerte:

$E_l + E_n = \frac{\hbar^2\pi^2}{2m}\left(\frac{l^2}{a^2} + \frac{n^2}{b^2}\right)$ (1)

Entartung:

$E_{n,l} - E_{n',l'} = 0 \Rightarrow \frac{\hbar^2\pi^2}{2m}\left(\frac{l^2 - l'^2}{a^2} + \frac{n^2 - n'^2}{b^2}\right) = 0$ (1)

$$\frac{l^2 - l'^2}{a^2} + \frac{n^2 - n'^2}{b^2} = 0$$

$$\frac{a^2}{b^2} = \frac{l^2 - l'^2}{n'^2 - n^2}$$

$$\left|\frac{a}{b}\right| = \sqrt{\frac{l^2 - l'^2}{n'^2 - n^2}}, \quad \frac{l^2 - l'^2}{n'^2 - n^2} \geq 0$$ (2)

(b) $\sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$ (1)

6. (8 Punkte)

(a)

$\Psi(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\Psi_0 \Rightarrow \frac{\partial}{\partial t}\Psi(t) = -\frac{\mathrm{i}}{\hbar}H\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\Psi_0 = -\frac{\mathrm{i}}{\hbar}H\Psi(t)$ (1)

$\Psi_0 = \sum_n c_n\phi_n, \quad H\phi_n = E_n\phi_n$ (2)

$\Psi(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\Psi_0 = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\sum_n c_n\phi_n = \sum_n c_n\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\phi_n = \sum_n c_n\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_nt}\phi_n$ (2)

weil

$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}\phi_n = \sum_{k=0}^{\infty}\frac{\left(-\frac{\mathrm{i}}{\hbar}Ht\right)^k}{k!}\phi_n = \sum_{k=0}^{\infty}\frac{\left(-\frac{\mathrm{i}}{\hbar}E_nt\right)^k}{k!}\phi_n = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_nt}\phi_n$

$c_n = \langle\phi_n \mid \Psi_0\rangle$ (1)

$\Psi(t) = \sum_n \langle\phi_n \mid \Psi_0\rangle\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_nt}\phi_n$ (1)

(b) $c_n^2 = \langle\phi_n|\Psi_0\rangle\,\langle\Psi_0|\phi_n\rangle = |\langle\phi_n|\Psi_0\rangle|^2$ (1)

7. (1 Punkt)

$$E_m - E_n = \hbar\omega\left(m+\tfrac{1}{2}\right) - \hbar\omega\left(n+\tfrac{1}{2}\right) = \hbar\omega\,(m-n) \qquad (1)$$

8. (11 Punkte)

$$\begin{array}{l} a = \frac{1}{\sqrt{2}}\left(\hat{\xi} + \mathrm{i}\hat{p}_\xi\right) \\ a^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\xi} - \mathrm{i}\hat{p}_\xi\right) \end{array} \Rightarrow \begin{array}{l} \hat{\xi} = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \\ \hat{p}_\xi = -\frac{\mathrm{i}}{\sqrt{2}}\left(a - a^\dagger\right) \end{array} \qquad (1)$$

$$\begin{array}{ll} \Delta\xi = \sqrt{\left\langle \hat{\xi}^2 \right\rangle - \left\langle \hat{\xi} \right\rangle^2} & \Delta x = \sqrt{\frac{\hbar}{m\omega}}\Delta\xi \\ \Delta p_\xi = \sqrt{\left\langle \hat{p}_\xi^2 \right\rangle - \left\langle \hat{p}_\xi \right\rangle^2} & \Delta p = \sqrt{m\omega\hbar}\,\Delta p_\xi \end{array} \qquad (1)$$

$$\begin{aligned} \left\langle \xi^2 \right\rangle &= \frac{1}{2}\Big(\left\langle \phi_n \left| aa \right| \phi_n \right\rangle + \left\langle \phi_n \left| a^\dagger a^\dagger \right| \phi_n \right\rangle \\ &\quad + \left\langle \phi_n \left| aa^\dagger \right| \phi_n \right\rangle + \left\langle \phi_n \left| a^\dagger a \right| \phi_n \right\rangle\Big) \qquad (1) \\ &= \frac{1}{2}\Big(\left\langle \phi_n \left| a\sqrt{n} \right| \phi_{n-1} \right\rangle + \left\langle \phi_n \left| a^\dagger\sqrt{n+1} \right| \phi_{n+1} \right\rangle \\ &\quad + \left\langle \phi_n \left| a\sqrt{n+1} \right| \phi_{n+1} \right\rangle + \left\langle \phi_n \left| a^\dagger\sqrt{n} \right| \phi_{n-1} \right\rangle\Big) \qquad (1) \\ &= \frac{1}{2}\Big(\sqrt{n}\sqrt{n-1}\langle \phi_n | \phi_{n-2} \rangle + \sqrt{n+1}\sqrt{n+2}\langle \phi_n | \phi_{n+2} \rangle \\ &\quad + (n+1)\left\langle \phi_n | \phi_n \right\rangle + n\langle \phi_n | \phi_n \rangle\Big) = n + \frac{1}{2} \qquad (1) \end{aligned}$$

$$\begin{aligned} \langle \xi \rangle &= \frac{1}{\sqrt{2}}\left(\left\langle \phi_n \left| a \right| \phi_n \right\rangle + \left\langle \phi_n \left| a^\dagger \right| \phi_n \right\rangle\right) \qquad (2) \\ &= \frac{1}{\sqrt{2}}\left(\sqrt{n}\langle \phi_n | \phi_{n-1} \rangle + \sqrt{n+1}\langle \phi_n | \phi_{n+1} \rangle\right) \\ &= 0 \end{aligned}$$

$$\langle p_\xi \rangle = -\frac{i}{\sqrt{2}}\left(\langle \phi_n | a | \phi_n \rangle - \langle \phi_n | a^\dagger | \phi_n \rangle\right) = 0 \qquad (1)$$

$$\left\langle p_\xi^2 \right\rangle = -\frac{1}{2}\left(\langle \phi_n | aa | \phi_n \rangle - \langle \phi_n | aa^\dagger | \phi_n \rangle - \langle \phi_n | a^\dagger a | \phi_n \rangle + \langle \phi_n | a^\dagger a^\dagger | \phi_n \rangle\right) = n + \frac{1}{2} \qquad (1)$$

$$\begin{aligned} \Delta x \Delta p &= \sqrt{\frac{\hbar}{m\omega}}\Delta\xi\sqrt{m\omega\hbar}\,\Delta p_\xi \qquad (1) \\ &= \hbar\Delta\xi\Delta p_\xi \\ &= \hbar\sqrt{n+\frac{1}{2}}\sqrt{n+\frac{1}{2}} \\ &= \hbar\left(n+\frac{1}{2}\right) \end{aligned}$$

$\Delta x \Delta p \geq \frac{\hbar}{2}$ für alle $n = 0, 1, \ldots$ (Heis. Unschärferelation) (1)

9. (5 Punkte)

(a)

$$\begin{array}{ll} l=0 & m=0 \\ l=1 & m=-1,0,1 \\ l=2 & m=-2,-1,0,1,2 \\ l=3 & m=-3,-2,-1,0,1,2,3 \end{array} \qquad (1)$$

(b)

$$E_m - E_n = -\frac{\mathrm{R}}{m^2} + \frac{\mathrm{R}}{n^2} = \mathrm{R}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \qquad (1)$$

(c)

(i)

$$2\hbar, -2\hbar \qquad (1)$$

(ii)

$$2\hbar: \quad \left(-\frac{\mathrm{i}}{\sqrt{2}}\right)\left(\frac{\mathrm{i}}{\sqrt{2}}\right) = \frac{1}{2} \qquad (1)$$

$$-2\hbar: \quad \left(\frac{\mathrm{i}}{\sqrt{2}}\right)\left(-\frac{\mathrm{i}}{\sqrt{2}}\right) = \frac{1}{2} \qquad (1)$$