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Blatt 1
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Übungen zur Vorlesung Mathematik für Chemiker 1

Lösungen

Aufgabe 1

$$
\begin{aligned}
(a) \sum_{k=1}^{n}(a_k - a_{k-1}) &= \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{k-1} \\
&\downarrow \\
&\downarrow [i = k-1] \\
&\downarrow \\
&= \sum_{k=1}^{n} a_k - \sum_{i=0}^{n-1} a_i \\
&= a_n - a_0
\end{aligned}
$$

$$
\begin{aligned}
(b) \sum_{j=1}^{2n} a_j &= a_1 + a_2 + a_3 + a_4 + ... + a_{2n-1} + a_{2n} \\
&= \sum_{j=1}^{n} a_{2j-1} + \sum_{j=1}^{n} a_{2j} \\
&= \sum_{k=0}^{n-1} a_{2k+1} + \sum_{j=1}^{n} a_{2j} \\
\sum_{j=1}^{2n} a_j - \sum_{k=0}^{n-1} a_{2k+1} &= \sum_{j=1}^{n} a_{2j}
\end{aligned}
$$

Aufgabe 2

Induktionsvoraussezung:
Die Behauptung gilt für n=1:

$$
\begin{aligned}
\sum_{k=1}^{1} k^2 &= 1 \\
\frac{1(1+1)(2 \times 1 + 1)}{6} &= \frac{6}{6} = 1
\end{aligned}
$$

Induktionsannahme:
Für eine natürliche Zahl $n \in \mathbb{N}$ gelte

$$
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}
$$

Induktionsschluss

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n+1}{6}[n(2n+1) + 6(n+1)] \\
&= \frac{n+1}{6}[2n^2 + 7n + 6] \\
&= \frac{1}{6}(n+1)(n+2)(2n+3) \\
&= \frac{1}{6}(n+1)[(n+1)+1][2(n+1)+1]
\end{aligned}$$

Aufgabe 3

$$\begin{aligned}
(a) \sum_{k=1}^{5}(3k-9) &= \sum_{k=1}^{5} 3k - \sum_{k=1}^{5} 9 \\
&= 3\sum_{k=1}^{5} k - 5 \times 9 \\
&= 3 \times \frac{5 \times 6}{2} - 45 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(b) \sum_{j=1}^{5} j(j-1) &= \sum_{j=1}^{5}(j^2 - j) \\
&= \sum_{j=1}^{5} j^2 - \sum_{j=1}^{5} j \\
&= \frac{1}{6} \times 5 \times (5+1) \times (2 \times 5 + 1) - \frac{5 \times 6}{2} \\
&= 40
\end{aligned}$$

Aufgabe 4

(a) Induktionsanfang: Die Behauptung gilt fÃ¼r $n = 1$

$$\sum_{k=1}^{1} \frac{1}{4k^2-1} = \frac{1}{4-1} = \frac{1}{3}$$

$$\frac{1}{2 \cdot 1 + 1} = \frac{1}{3}$$

Induktionsannahme: FÃ¼r eine natÃ¼rliche Zahl $n \in \mathbb{N}$ gelte

$$\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$$

Induktionsschluss:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{4k^2-1} &= \sum_{k=1}^{n} \frac{1}{4k^2-1} + \frac{1}{4(n+1)^2-1} \\
&= \frac{n}{2n+1} + \frac{1}{4n^2+8n+3} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{1}{2n+1}[n + \frac{1}{2n+3}] \\
&= \frac{1}{2n+1}[\frac{2n^2+3n+1}{2n+3}] \\
&= \frac{1}{2n+1}[\frac{(n+1)(2n+1)}{2n+3}] \\
&= \frac{n+1}{2n+3} \\
&= \frac{n+1}{2(n+1)+1}
\end{aligned}$$

(b) Induktionsanfang: Die Behauptung gilt fÃ¼r $n = 1$

$$\prod_{k=1}^{1} \frac{k+1}{k} = 2$$

$$1 + 1 = 2$$

Induktionsannahme: FÃ¼r eine natÃ¼rliche Zahl $n \in \mathbb{N}$ gelte

$$\prod_{k=1}^{n} \frac{k+1}{k} = n+1$$

Induktionsschluss:

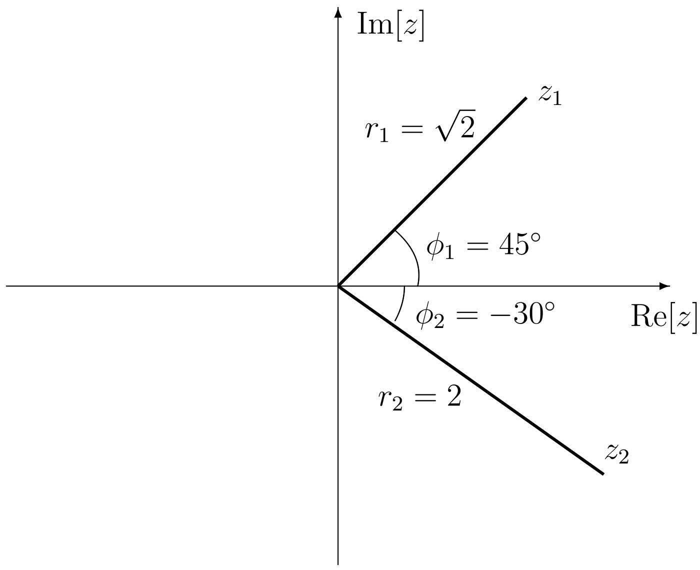
$$\begin{aligned}
\prod_{k=1}^{n+1} \frac{k+1}{k} &= (\prod_{k=1}^{n} \frac{k+1}{k})\frac{n+2}{n+1} \\
&= (n+1)\frac{n+2}{n+1} \\
&= n+2 \\
&= (n+1)+1
\end{aligned}$$

Komplexe Zahlen
Aufgabe 5

$$\begin{aligned}
&(a) \quad \frac{2+i}{1-2i} = \frac{(2+i)(1+2i)}{(1-2i)(1+2i)} = \frac{5i}{5} = i \\
&(b) \quad \frac{(1+i)(2-i)}{1-i} = \frac{(1+i)^2(2-i)}{(1-i)(1+i)} = \frac{2i(2-i)}{2} = 1+2i \\
&(c) \quad \frac{(1+2i)^2}{2+3i} = \frac{(1+2i)^2(2-3i)}{(2+3i)(2-3i)} = \frac{(-3+4i)(2-3i)}{13} = \frac{6}{13} + \frac{17}{13}i
\end{aligned}$$

Aufgabe 6

a) Gegeben sind: $z_1 = 1 + i = r_1 \mathrm{e}^{i\phi_1}$ und $z_2 = \sqrt{3} - i = r_2 \mathrm{e}^{i\phi_2}$.



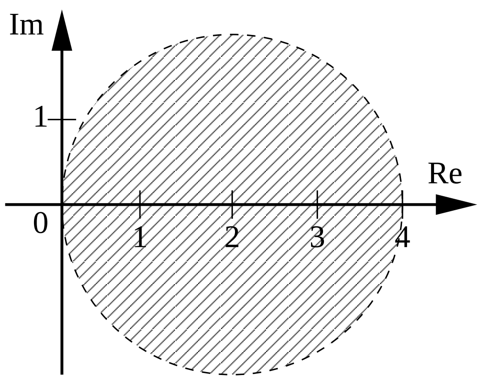
c)

$$\begin{aligned}
z_1 \cdot z_2 &= (1+i)(\sqrt{3}-i) = \sqrt{3}+1+(\sqrt{3}-1)i \\
z_1 \cdot z_2^* &= (1+i)(\sqrt{3}+i) = \sqrt{3}-1+(\sqrt{3}+1)i \\
|z_1 \cdot z_2| &= |(1+i)||(\sqrt{3}-i)| = 2\sqrt{2} \\
|z_1 \cdot z_2^*| &= |(1+i)||(\sqrt{3}+i)| = 2\sqrt{2} \\
\frac{z_1}{z_2} &= \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{\sqrt{3}-1}{4} + \frac{\sqrt{3}+1}{4} i
\end{aligned}$$

Aufgabe 7

a)

$$\begin{aligned}
&|z-2| < 2 \qquad \text{mit} \qquad z = x + iy \\
&|z-2| = |x-2+iy| = \sqrt{(x-2)^2+y^2} \\
&\quad \Leftrightarrow 0 \leq \sqrt{(x-2)^2+y^2} < 2
\end{aligned}$$



Offene Kreisschreibe um den Punkt 2 mit Radius 2

Aufgabe 8

a)

$$z_2 = \sqrt{3} - i = |z_2| \exp{(i\phi_2)}$$
$$|z_2| = 2$$
$$z_2 = |z_2|(\cos\phi_2 + i\sin\phi_2)$$
$$\sqrt{3} = 2\cos\phi_2, -1 = 2\sin\phi_2$$
$$\cos\phi_2 = \frac{\sqrt{3}}{2}, \sin\phi_2 = -\frac{1}{2}$$
$$\phi_2 = \frac{11}{6}\pi$$
$$z_2 = 2\exp{(i\frac{11}{6}\pi)}$$

$$z_3 = 1 + i\sqrt{3} = |z_3| \exp{(i\phi_3)}$$
$$|z_3| = 2$$
$$z_3 = |z_3|(\cos\phi_3 + i\sin\phi_3)$$
$$\cos\phi_3 = \frac{1}{2}, \sin\phi_2 = \frac{\sqrt{3}}{2}$$
$$\phi_3 = \frac{1}{3}\pi$$
$$z_3 = 2\exp{(i\frac{1}{3}\pi)}$$

$$z_2^* = \sqrt{3} + i$$
$$|z_2^*| = 2$$
$$\cos\phi_2 = \frac{\sqrt{3}}{2}, \sin\phi_2 = \frac{1}{2}$$
$$\phi_2 = \frac{1}{6}\pi$$
$$z_2^* = 2\exp{(i\frac{1}{6}\pi)}$$

$$z_3^* = 1 - i\sqrt{3}$$
$$|z_3^*| = 2$$
$$\cos\phi_3 = \frac{1}{2}, \sin\phi_2 = -\frac{\sqrt{3}}{2}$$
$$\phi_3 = \frac{5}{3}\pi$$
$$z_3^* = 2\exp{(i\frac{5}{3}\pi)}$$

b)

$$\begin{aligned}
z_2 z_2^* &= 2\exp{(i\frac{11}{6}\pi)}2\exp{(i\frac{1}{6}\pi)} \\
&= 4\exp{(i\frac{11}{6}\pi + i\frac{1}{6}\pi)} \\
&= 4\exp{(2i\pi)} \\
&= 4\exp{(0)} \\
&= 4 \\
&= |z_2|^2
\end{aligned}$$

$$\begin{aligned}
z_3 z_3^* &= 2\exp{(i\frac{1}{3}\pi)}2\exp{(i\frac{5}{3}\pi)} \\
&= 4\exp{2i\pi} \\
&= 4\exp{0} \\
&= 4 \\
&= |z_3|^2
\end{aligned}$$

c)

$$\begin{aligned}
z_2^4 &= (2\exp{(i\frac{11}{6}\pi)})^4 \\
&= 2^4\exp{(i4\frac{11}{6}\pi)} \\
&= 16\exp{(i\frac{22}{3}\pi)} \\
&= 16\exp{[i(\frac{4}{3}\pi + 6\pi)]} \\
&= 16\exp{(i\frac{4}{3}\pi)}
\end{aligned}$$

$$\begin{aligned}
z_3^4 &= (2\exp{(i\frac{1}{3}\pi)})^4 \\
&= 16\exp{(i\frac{4}{3}\pi)} \\
&= z_2^4 \qquad (1)
\end{aligned}$$

Anm:

$$\begin{aligned}
z_3 &= 1 + i\sqrt{3} \\
&= i(\sqrt{3} - i) \\
&= iz_2 \\
z_3^4 &= (iz_2)^4 = i^4 z_2^4 = z_2^4 \qquad (2)
\end{aligned}$$

Aufgabe 9

(c)

$$z = x + iy = r\exp(i\phi) = r\cos\phi + ir\sin\phi$$
$$x = r\cos\phi, y = r\sin\phi, r = \sqrt{x^2+y^2} = |z|, \tan\phi = \frac{y}{x}$$

$$z = -1 - i$$
$$r = \sqrt{1+1} = \sqrt{2}$$
$$-1 = \sqrt{2}\cos\phi, -1 = \sqrt{2}\sin\phi$$
$$\cos\phi = -\frac{1}{\sqrt{2}}, \sin\phi = -\frac{1}{\sqrt{2}}$$
$$\phi = \frac{5}{4}\pi$$
$$z = \sqrt{2}\exp(i\frac{5}{4}\pi)$$

(b)

$$z = \frac{2}{1-i}$$
$$\tilde{z} = 1 - i$$
$$\tilde{r} = \sqrt{1+1} = \sqrt{2}$$
$$1 = \sqrt{2}\cos\phi, -1 = \sqrt{2}\sin\phi$$
$$\cos\phi = \frac{1}{\sqrt{2}}, \sin\phi = -\frac{1}{\sqrt{2}}$$
$$\phi = \frac{7}{4}\pi$$
$$\tilde{z} = \sqrt{2}\exp(i\frac{7}{4}\pi)$$
$$z = \frac{2}{\tilde{z}} = \frac{2}{\sqrt{2}\exp(i\frac{7}{4}\pi)} = \sqrt{2}\exp(-i\frac{7}{4}\pi) = \sqrt{2}\exp(i2\pi - i\frac{7}{4}\pi) = \sqrt{2}\exp(i\frac{1}{4}\pi)$$

(a)

$$\begin{aligned} z &= (1-i)^7 \\ &= (\sqrt{2}\exp(i\frac{7}{4}\pi))^7 \\ &= (\sqrt{2})^7\exp(i\frac{49}{4}\pi) \\ &= 8\sqrt{2}\exp[i(\frac{1}{4}\pi + 12\pi)] \\ &= 8\sqrt{2}\exp(i\frac{1}{4}\pi) \end{aligned}$$

Aufgabe 10

(a) Gesucht sind alle LÃ¶sungen von $z^2 = \sqrt{3}+i$, d.h. alle zweiten Wurzeln von $\xi = \sqrt{3}+i = z^2$

$$\xi = \sqrt{3} + i = 2\exp\left(i\frac{\pi}{6}\right)$$

wegen

$$
\begin{aligned}
r &= |\xi| = \sqrt{3+1^2} = 2 \\
\Re\xi &= \sqrt{3} = r\cos\varphi = 2\cos\varphi \quad \Rightarrow \quad \cos\varphi = \frac{\sqrt{3}}{2} \\
\Im\xi &= 1 = r\sin\varphi = 2\sin\varphi \quad \Rightarrow \quad \sin\varphi = \frac{1}{2} \\
&\Longrightarrow \quad \varphi = \frac{\pi}{6}
\end{aligned}
$$

Allgemein sind die n-ten Wurzeln von $\xi = r\exp(i\varphi)$ gegeben durch $(k = 0, 1, 2..., n-1)$:

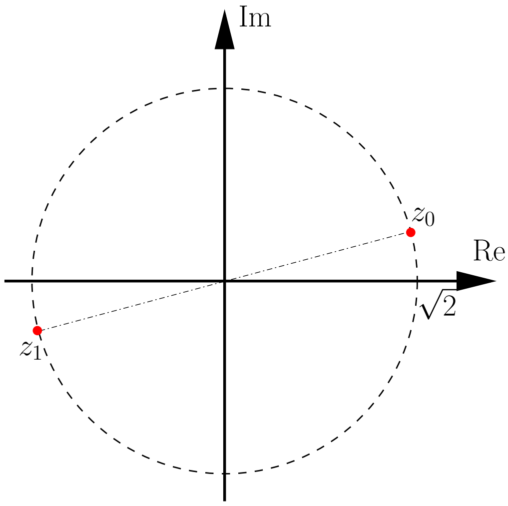
$$
z_k = \sqrt[n]{r}\exp(i\varphi/n + 2\pi ik/n) = \sqrt[n]{r}\big[\cos(\varphi/n + 2\pi k/n) + i\sin(\varphi/n + 2\pi k/n)\big]
$$

Die 2 LÃ¶sungen von $z^2 = \xi$ sind $(k = 0, 1)$:

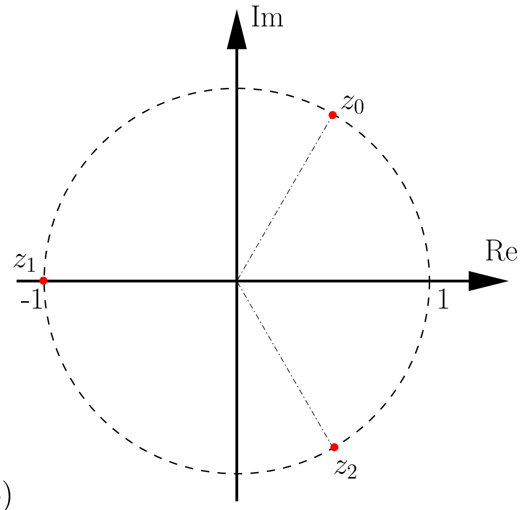
$$
\begin{aligned}
z_k &= \sqrt{2}\exp\big[\frac{i\pi}{12} + i\pi k\big] = \sqrt{2}\exp\big[i\pi(\frac{1}{12} + k)\big] = \sqrt{2}\big[\cos\big(\pi(\frac{1}{12}+k)\big) + i\sin\big(\pi(\frac{1}{12}+k)\big)\big] \\
z_0 &= \sqrt{2}\big[\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12})\big] = 1.366 + 0.366i \\
z_1 &= \sqrt{2}\big[\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})\big] = -1.366 - 0.366i = -z_0
\end{aligned}
$$

(b) Gesucht sind alle dritten Wurzeln von $\xi = z^3 = -1 = \exp(i\pi)$, also

$$
\begin{aligned}
z_k &= \exp\big(i\frac{1}{3}\pi + 2ik\pi\frac{1}{3}\big) = \cos\big(\frac{\pi}{3} + \frac{2k\pi}{3}\big) + i\sin\big(\frac{\pi}{3} + \frac{2k\pi}{3}\big), \quad k = 0, 1, 2 \\
z_0 &= \cos\big(\frac{\pi}{3}\big) + i\sin\big(\frac{\pi}{3}\big) = \frac{1}{2} + \frac{\sqrt{3}i}{2} \\
z_1 &= \cos(\pi) + i\sin(\pi) = -1 \\
z_2 &= \cos\big(\frac{5\pi}{3}\big) + i\sin\big(\frac{5\pi}{3}\big) = \frac{1}{2} - \frac{\sqrt{3}i}{2} = z_0^*
\end{aligned}
$$



(a)



(b)