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SoSe 2013
Blatt 1
10-17.4.2013

Übungen zur Vorlesung Mathematik für Chemiker 2

Lösungen

Komplexe Zahlen
Aufgabe 1

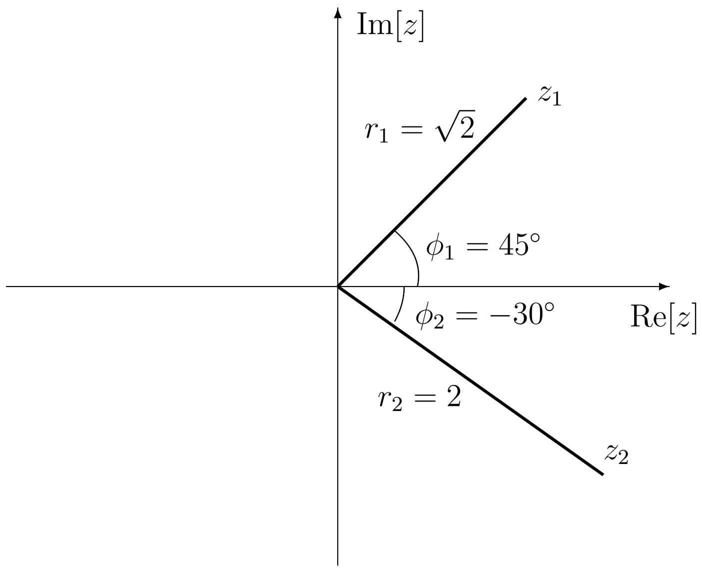
$$(a)\qquad \frac{2+i}{1-2i} = \frac{(2+i)(1+2i)}{(1-2i)(1+2i)} = \frac{5i}{5} = i$$

$$(b)\qquad \frac{(1+i)(2-i)}{1-i} = \frac{(1+i)^2(2-i)}{(1-i)(1+i)} = \frac{2i(2-i)}{2} = 1+2i$$

$$(c)\qquad \frac{(1+2i)^2}{2+3i} = \frac{(1+2i)^2(2-3i)}{(2+3i)(2-3i)} = \frac{(-3+4i)(2-3i)}{13} = \frac{6}{13} + \frac{17}{13}i$$

Aufgabe 2

a) Gegeben sind: $z_1 = 1 + i = r_1 e^{i\phi_1}$ und $z_2 = \sqrt{3} - i = r_2 e^{i\phi_2}$.



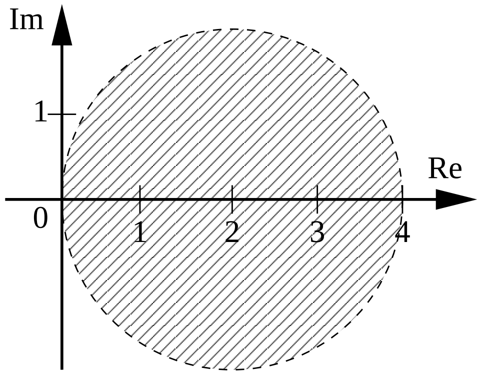
c)

$$z_1 \cdot z_2 = (1+i)(\sqrt{3}-i) = \sqrt{3}+1+(\sqrt{3}-1)i$$
$$z_1 \cdot z_2^* = (1+i)(\sqrt{3}+i) = \sqrt{3}-1+(\sqrt{3}+1)i$$
$$|z_1 \cdot z_2| = |(1+i)||(\sqrt{3}-i)| = 2\sqrt{2}$$
$$|z_1 \cdot z_2^*| = |(1+i)||(\sqrt{3}+i)| = 2\sqrt{2}$$
$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{\sqrt{3}-1}{4} + \frac{\sqrt{3}+1}{4}i$$

Aufgabe 3

a)

$$|z-2| < 2 \qquad \text{mit} \qquad z = x+iy$$
$$|z-2| = |x-2+iy| = \sqrt{(x-2)^2+y^2}$$
$$\Leftrightarrow 0 \leq \sqrt{(x-2)^2+y^2} < 2$$



Offene Kreisschreibe um den Punkt 2 mit Radius 2

Aufgabe 4

a)

$$z_2 = \sqrt{3} - i = |z_2| \exp{(i\phi_2)}$$
$$|z_2| = 2$$
$$z_2 = |z_2|(\cos\phi_2 + i \sin\phi_2)$$
$$\sqrt{3} = 2\cos\phi_2, -1 = 2\sin\phi_2$$
$$\cos\phi_2 = \frac{\sqrt{3}}{2}, \sin\phi_2 = -\frac{1}{2}$$
$$\phi_2 = \frac{11}{6}\pi$$
$$z_2 = 2\exp{(i\frac{11}{6}\pi)}$$

$$z_3 = 1 + i\sqrt{3} = |z_3| \exp{(i\phi_3)}$$
$$|z_3| = 2$$
$$z_3 = |z_3|(\cos\phi_3 + i \sin\phi_3)$$
$$\cos\phi_3 = \frac{1}{2}, \sin\phi_2 = \frac{\sqrt{3}}{2}$$
$$\phi_3 = \frac{1}{3}\pi$$
$$z_3 = 2\exp{(i\frac{1}{3}\pi)}$$

$$z_2^* = \sqrt{3} + i$$
$$|z_2^*| = 2$$
$$\cos\phi_2 = \frac{\sqrt{3}}{2}, \sin\phi_2 = \frac{1}{2}$$
$$\phi_2 = \frac{1}{6}\pi$$
$$z_2^* = 2\exp{(i\frac{1}{6}\pi)}$$

$$z_3^* = 1 - i\sqrt{3}$$
$$|z_3^*| = 2$$
$$\cos\phi_3 = \frac{1}{2}, \sin\phi_2 = -\frac{\sqrt{3}}{2}$$
$$\phi_3 = \frac{5}{3}\pi$$
$$z_3^* = 2\exp{(i\frac{5}{3}\pi)}$$

b)

$$\begin{aligned}
z_2 z_2^* &= 2\exp{(i\frac{11}{6}\pi)} 2\exp{(i\frac{1}{6}\pi)} \\
&= 4\exp{(i\frac{11}{6}\pi + i\frac{1}{6}\pi)} \\
&= 4\exp{(2i\pi)} \\
&= 4\exp{(0)} \\
&= 4 \\
&= |z_2|^2
\end{aligned}$$

$$\begin{aligned}
z_3 z_3^* &= 2\exp{(i\frac{1}{3}\pi)} 2\exp{(i\frac{5}{3}\pi)} \\
&= 4\exp{2i\pi} \\
&= 4\exp{0} \\
&= 4 \\
&= |z_3|^2
\end{aligned}$$

c)

$$\begin{aligned}
z_2^4 &= (2\exp{(i\frac{11}{6}\pi)})^4 \\
&= 2^4\exp{(i4\frac{11}{6}\pi)} \\
&= 16\exp{(i\frac{22}{3}\pi)} \\
&= 16\exp{[i(\frac{4}{3}\pi + 6\pi)]} \\
&= 16\exp{(i\frac{4}{3}\pi)}
\end{aligned}$$

$$\begin{aligned}
z_3^4 &= (2\exp{(i\frac{1}{3}\pi)})^4 \\
&= 16\exp{(i\frac{4}{3}\pi)} \\
&= z_2^4 \qquad (1)
\end{aligned}$$

Anm:

$$\begin{aligned} z_3 &= 1+i\sqrt{3} \\ &= i(\sqrt{3}-i) \\ &= iz_2 \\ z_3^4 &= (iz_2)^4 = i^4 z_2^4 = z_2^4 \end{aligned} \tag{2}$$

Aufgabe 5

(c)

$$z = x + iy = r\exp(i\phi) = r\cos\phi + ir\sin\phi$$
$$x = r\cos\phi, y = r\sin\phi, r = \sqrt{x^2+y^2} = |z|, \tan\phi = \frac{y}{x}$$

$$z = -1 - i$$
$$r = \sqrt{1+1} = \sqrt{2}$$
$$-1 = \sqrt{2}\cos\phi, -1 = \sqrt{2}\sin\phi$$
$$\cos\phi = -\frac{1}{\sqrt{2}}, \sin\phi = -\frac{1}{\sqrt{2}}$$
$$\phi = \frac{5}{4}\pi$$
$$z = \sqrt{2}\exp(i\frac{5}{4}\pi)$$

(b)

$$z = \frac{2}{1-i}$$
$$\tilde{z} = 1 - i$$
$$\tilde{r} = \sqrt{1+1} = \sqrt{2}$$
$$1 = \sqrt{2}\cos\phi, -1 = \sqrt{2}\sin\phi$$
$$\cos\phi = \frac{1}{\sqrt{2}}, \sin\phi = -\frac{1}{\sqrt{2}}$$
$$\phi = \frac{7}{4}\pi$$
$$\tilde{z} = \sqrt{2}\exp(i\frac{7}{4}\pi)$$
$$z = \frac{2}{\tilde{z}} = \frac{2}{\sqrt{2}\exp(i\frac{7}{4}\pi)} = \sqrt{2}\exp(-i\frac{7}{4}\pi) = \sqrt{2}\exp(i2\pi - i\frac{7}{4}\pi) = \sqrt{2}\exp(i\frac{1}{4}\pi)$$

(a)

$$\begin{aligned} z &= (1-i)^7 \\ &= (\sqrt{2}\exp(i\frac{7}{4}\pi))^7 \\ &= (\sqrt{2})^7\exp(i\frac{49}{4}\pi) \\ &= 8\sqrt{2}\exp[i(\frac{1}{4}\pi + 12\pi)] \\ &= 8\sqrt{2}\exp(i\frac{1}{4}\pi) \end{aligned}$$

Aufgabe 6

(a) Gesucht sind alle Lösungen von $z^2 = \sqrt{3}+i$, d.h. alle zweiten Wurzeln von $\xi = \sqrt{3}+i = z^2$

$$\xi = \sqrt{3}+i = 2\exp\big(i\frac{\pi}{6}\big)$$

wegen

$$r = |\xi| = \sqrt{3+1^2} = 2$$
$$\Re\xi = \sqrt{3} = r\cos\varphi = 2\cos\varphi \quad \Rightarrow \quad \cos\varphi = \frac{\sqrt{3}}{2}$$
$$\Im\xi = 1 = r\sin\varphi = 2\sin\varphi \quad \Rightarrow \quad \sin\varphi = \frac{1}{2}$$
$$\Longrightarrow \quad \varphi = \frac{\pi}{6}$$

Allgemein sind die n-ten Wurzeln von $\xi = r\exp(i\varphi)$ gegeben durch ($k = 0, 1, 2..., n-1$):

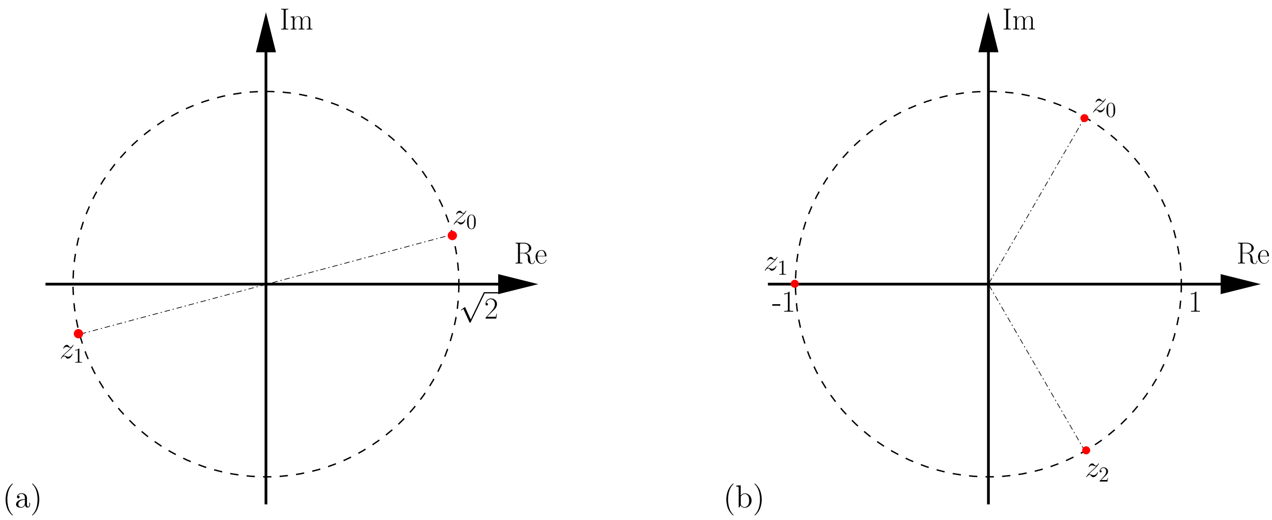
$$z_k = \sqrt[n]{r}\exp(i\varphi/n + 2\pi ik/n) = \sqrt[n]{r}\big[\cos(\varphi/n + 2\pi k/n) + i\sin(\varphi/n + 2\pi k/n)\big]$$

Die 2 Lösungen von $z^2 = \xi$ sind ($k = 0, 1$):

$$z_k = \sqrt{2}\exp\big[\frac{i\pi}{12} + i\pi k\big] = \sqrt{2}\exp\big[i\pi(\frac{1}{12}+k)\big] = \sqrt{2}\big[\cos\big(\pi(\frac{1}{12}+k)\big) + i\sin\big(\pi(\frac{1}{12}+k)\big)\big]$$
$$z_0 = \sqrt{2}\big[\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12})\big] = 1.366 + 0.366i$$
$$z_1 = \sqrt{2}\big[\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})\big] = -1.366 - 0.366i = -z_0$$

(b) Gesucht sind alle dritten Wurzeln von $\xi = z^3 = -1 = \exp(i\pi)$, also

$$z_k = \exp\big(i\frac{1}{3}\pi + 2ik\pi\frac{1}{3}\big) = \cos\big(\frac{\pi}{3} + \frac{2k\pi}{3}\big) + i\sin\big(\frac{\pi}{3} + \frac{2k\pi}{3}\big), \quad k = 0, 1, 2$$
$$z_0 = \cos\big(\frac{\pi}{3}\big) + i\sin\big(\frac{\pi}{3}\big) = \frac{1}{2} + \frac{\sqrt{3}i}{2}$$
$$z_1 = \cos(\pi) + i\sin(\pi) = -1$$
$$z_2 = \cos\big(\frac{5\pi}{3}\big) + i\sin\big(\frac{5\pi}{3}\big) = \frac{1}{2} - \frac{\sqrt{3}i}{2} = z_0^*$$



Vektorräume

Aufgabe 7

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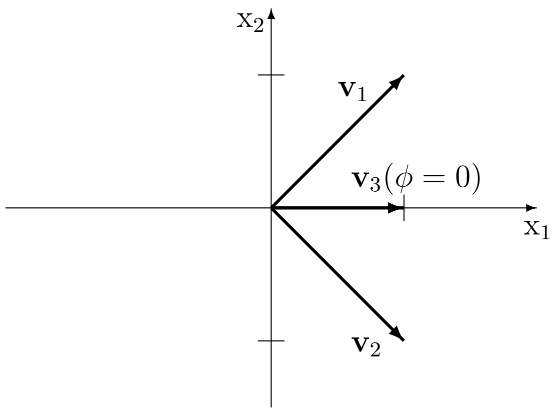
Aufgabe 8

Lineare Abhängigkeit: bestimme ϕ derart, dass:

$$\mathbf{v}_3 = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \qquad \text{mit } \lambda_1, \lambda_2 \in \mathbb{R},$$

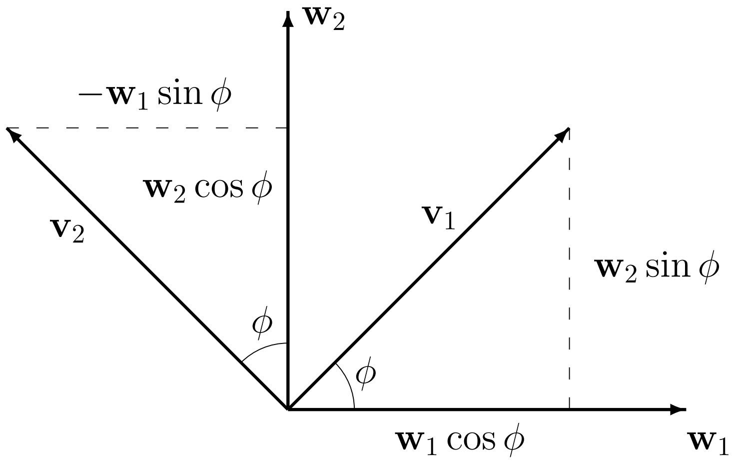
$$\Rightarrow \quad \begin{pmatrix} 1 \\ 0 \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \\ 0 \end{pmatrix},$$

Offensichtlich erfüllt für $\phi = 0, \pi, 2\pi, \ldots$, sowie $\lambda_1 = \lambda_2 = \frac{1}{2}$.
Lineare Abhängigkeit $\Leftrightarrow$ alle 3 Vektoren liegen in einer Ebene:



Aufgabe 9

a) $\mathbf{w}_1$ und $\mathbf{w}_2$ sind orthogonal und normiert (bzw. orthonormiert), d.h. wir können zeichnen:



b)

$$\phi = \frac{\pi}{4}, \quad \rightarrow \quad \sin\phi = \frac{1}{\sqrt{2}} = \cos\phi,$$

$$\rightarrow \qquad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{w}_1 + \mathbf{w}_2\right), \quad (1)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\left(-\mathbf{w}_1 + \mathbf{w}_2\right), \quad (2)$$

$$(1)-(2): \qquad \mathbf{v}_1 - \mathbf{v}_2 = \sqrt{2}\mathbf{w}_1, \quad \rightarrow \quad \mathbf{w}_1 = \frac{1}{\sqrt{2}}\mathbf{v}_1 - \frac{1}{\sqrt{2}}\mathbf{v}_2,$$

$$(1)+(2): \qquad \mathbf{v}_1 + \mathbf{v}_2 = \sqrt{2}\mathbf{w}_2, \quad \rightarrow \quad \mathbf{w}_2 = \frac{1}{\sqrt{2}}\mathbf{v}_1 + \frac{1}{\sqrt{2}}\mathbf{v}_2,$$

$$\Rightarrow \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

c)

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2 = \frac{1}{\sqrt{2}}\left[(\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_1 + \mathbf{v}_2)\right] = \sqrt{2}\mathbf{v}_1.$$